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# Prey-predator Eco-epidemiological Model with Nonlinear Transmission of Disease

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**ABSTRACT**

**Nonlinear transmission of disease between infected and uninfected prey was studied using a prey-predator eco-epidemiological model. The interaction of predators with infected and uninfected prey species depends on their numerical superiority. Harvesting of both uninfected and infected prey was carried out, and stability analysis was carried out for equilibrium values. Using the parameter $\mu$, the death rate of infected prey as a bifurcation parameter, it is shown that Hopf bifurcation could occur. The theoretical results are compared with numerical results for different sets of parameters.**

**Keywords: Prey, predator, Hopf bifurcation, harvesting, stability**

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## INTRODUCTION

In eco-epidemiology ecological and epidemiological topics are studied simultaneously (Chattopadhyay and Arino,1999; Chattopadhyay and Bairagi, 2001; Hadeler and Freeedman, 1989; Venturino, 2002), and a combination of an epidemic model and an ecological model is called an eco-epidemiological model. There has been great interest over the last few decades three different types of predator-prey interaction (a) where the prey population is infected, (b) where the predator population is infected and (c) where both predator and prey populations are infected. It has been verified experimentally that infectious diseases, regulate animal as well as human populations. This has become an interesting area of research, and mathematical models have been widely used to understand and analyze the spread and control of infectious diseases. Hseih and Hsiao, (2008) in a study of predator-prey interaction where both species were infected, found conditions under which a disease is wiped out from an eco-system and both species survive. Under normal conditions, if a predator feeds on infected prey extensively, then the predator population is driven to extinction. However, they discovered threshold level at which disease would become endemic in an ecosystem. Packer et al. (2003) determined that infection in a prey population is reduced by predation. After removing predators from different systems, Sih et al. (1985) proved that in 54 out of 135 systems, the proportion of prey infected by diseases declined. Findings by Hudson, (1992) also confirmed same that predator populations help maintain the prey population. Das et al. (2009) discussed a model where both predator and prey populations were infected by the same and different viral diseases and the predator population died not become infected by consuming infected prey. This was due to enzymatic activity and prey cells becoming incapable of transferring the infection to the predator population. They found that by controlling the predation of susceptible and infected prey, a system can become disease free.

In a predator-prey environment, if the disease spreads among prey, predators can feed on either uninfected prey or infected prey, which may become either easier or more difficult to capture by predators (Holmes and Bethel, 1972; Murray et al., 1997). There are biological reasons for assuming both situations, firstly since the hunting of diseased, i.e., weaker individuals, makes their capture easier. On the other hand, the nutritional value of uninfected prey is certainly higher than of infected prey. The ingestion of diseased prey can have a negative impact on the predator population. Many field studies and experiments confirm that predators consume a disproportionately large number of infected preys (Hudson et a., 1992; Cohn, 2002; Moore,2002;Schaller, 1972). Infected prey often lives close to the water surface or on top of vegetation for more oxygen and hence become more vulnerable to predation (Frind, 2002; Kaiser and Salton, 1999). The presence of parasites in prey populations can enable both predator and prey populations to coexist. Predators cannot sustain stable populations in the absence of the parasite because predators are able to more easily catch infected individuals when the prey species are weakened due to infection. Hence predator and prey population are able to coexist if there is a certain level of infection in the prey population (Freedman et al., 1987).

Consumption of infected prey in the predators' diet has often been high because predators can capture them with less effort. When the infected prey population starts declining, then predators are forced to switch towards other types of susceptible prey. The switching feeding behaviour of predators with more than one source of food has been discussed by many researchers (Tansky, 1978; Mukhopadhyay and Bhattacharyya, 2009; Khan et al.,1998,2004; Hotopp et al.,2010). However, here, it is reconsidered in the context of eco-epidemiology. Experimental studies have verified that predation of infected prey is reduced if they are treated (Mukhopadhyay and Bhattacharyya, 2009). Several authors studying eco-epidemiological mathematical models have described predators switching abundant prey populations, easily catchable infected prey populations, infected prey refuge, ratio dependent functional responses, and external sources of disease (Haque and Venturino, 2006; Pal and Samanta, 2010; Kundu, 2006;Greenhalgh and Haque, 2007;Haque and Venturino, 2008,2009). In all these models, the authors assumed that the infection affects the prey population only and the disease transmission follows the simple law of mass action with a constant rate of transmission. They also assumed that predators consume either uninfected prey or infected prey. Haque and Greenhalgh, (2010), on the other hand, examined a predator-prey model where disease spreads among the prey population and predators were found not to consume the infected prey.

In line with above research, this study focuses the rate of predation on both susceptible and infected prey species. To the best of my knowledge, all authors, except (Naji and Mustafa, 2012) consider disease transmits according to the simple law of mass action with a constant rate of transmission. Naji and Mustafa, (2012) examined an eco-epidemiological model with a nonlinear incidence rate and predator switching. The model we consider here differs from this and other previous models in three ways.

(i) Measuring non-linear disease transmission and the inhibition effect on the behaviour of prey when there is an increase in the number of infected individuals.

(ii) Switching by the predator to uninfected and infected prey depending on numerical superiority. In our model, switching by predators is stronger than that discussed by Naji and Mustafa, (2012).

(iii) Harvesting of both uninfected and infected prey species.

## The model and assumptions

A predator-prey interaction is studied where the prey species breeds logistically in the absence of a predator, and it is represented mathematically by

$$rS\left(1-\frac{S+I}{k}\right)$$

Where $r$ is the growth rate of prey species, $S$ is the population density of uninfected prey species, and $I$ represent the population density of infected prey species. The uninfected and infected prey species share the same sea or pasture and compete for the same resources, so $k$ is the environment carrying capacity of prey species. $n$ and $m$ are the hunting efficiency of the predator towards uninfected and infected prey species respectively. Predator capture disease preys easily because due to the disease, they become inactive and so we are assuming $m > n$. We are assuming that only uninfected prey species have the capability of reproducing while infected species either do not have the capability of reproducing or die before attaining this age. When uninfected prey species come in contact with infected prey species, they become infected, according to a non-linear incidence rate of the form

$$\frac{\lambda SI}{a+\alpha I}\ [32],\ \text{where}\ \frac{1}{a+\alpha I}$$

denote inhibition effect of the behavioural change of uninfected prey when they come in contact with infected prey and $\lambda I$ represent the infection force of the disease. $a$ is the half saturation constant, and the parameter $\alpha$ represents the predator preference rate of $I$. The terms

$$\frac{nS^2P}{I+S}=\frac{nSP}{1+\frac{I}{S}}\ \text{and}\ \frac{mI^2P}{I+S}=\frac{mIP}{1+\frac{S}{I}}$$

have the property of switching, that is, if the population of uninfected prey species is higher than infected prey species, then interaction between predator and infected prey species will be high vice versa if the population of uninfected prey species is higher than infected prey species. Hence the predator will switch to the kind of prey species having numerical superiority. Zero denominator in switching terms is meaningless, and we can tell when the prey population will disappear the predator population will also vanish. Species are discrete and can be treated as zero if their densities become very small. We are assuming that conversion factor *n* and *m* represent the number of newly born predators for capturing each uninfected and infected prey species and these conversion factors are same as hunting efficiency. The interaction on infected prey will contribute positively to the diet of the predator though protein intake will be less than what predator will get by feeding uninfected prey. Since infected prey is toxinicated so by feeding infected prey, there will be an increase in the mortality rate of the predators which is denoted by *d*. $\mu$ is the death rate of the infected prey species. $q_1$ and $q_2$ are the catch-ability rates of catching uninfected and infected prey species respectively, where $q_1 < q_2$ since it is much easier to catch infected prey species than uninfected prey species because due to infection, infected prey become less active. *E* is the harvesting effort. We assume that all the parameters in the model are positives and that $S(0) > 0$, $I(0) > 0$, $P \geq (0)$. Taking into account the above assumptions the basic mathematical model is of the form:

$$\frac{dS}{dt} = rS\left(1-\frac{S+I}{k}\right) - \frac{\lambda SI}{a+\alpha I} - \frac{nS^2P}{I+S} - q_1ES,$$

$$\frac{dI}{dt} = \frac{\lambda SI}{a+\alpha I} - \frac{mI^2P}{I+S} - \mu I - q_2EI, \qquad (2.1)$$

$$\frac{dP}{dt} = \frac{nS^2P}{I+S} + \frac{mI^2P}{I+S} - dP$$

The solution of equations (2.1), will exist and unique within the region $S > 0$, $I > 0$ and $P \geq 0$ since the equations are continuously differentiable within this region (Arnold, 1971).

**Boundedness**

**Theorem 1**: The uninfected and infected prey is always bounded above. Proof from first two equations of the system (2.1), we get

$$\frac{dS}{dt} + \frac{dI}{dt} = rS\left(1-\frac{S+I}{k}\right) - \frac{S^2P}{I+S} - q_1ES - \mu I - q_2EI,$$

$$\frac{dS}{dt} + \frac{dI}{dt} < rS\left(1-\frac{S+I}{k}\right),$$

Where

$$\frac{dS}{dt} + \frac{dI}{dt} < kr,$$

If

$S + 1 < k$

Let $k = k_1 + k_2$

Considering $S < k_1$, and $I < k_2$.

Hence
$\lim_{t\to\infty} S(t) < k_1$,
and
$\lim_{t\to\infty} S(t) < k_2$,
So prey population is always bounded above

**Theorem 2:** *If q is the minimum of $q_2E$ and d then trajectories of the system (2.1) are bounded above.*

**Proof**

Let

$l = S + I + P$.

Take its derivative along the solution of (2.1), we get

$$\frac{dl}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dP}{dt}$$

$$\frac{dl}{dt} = rS\left(1-\frac{S+I}{k}\right) - q_1ES - \mu I - q_2EI - dP.$$

now,

$$\frac{dl}{dt} + ql = rS\left(1-\frac{S+I}{k}\right) - q_1ES - \mu I - q_2EI - dP + qS + qI + qP$$
$$< S(r+q) + I(q-q_2E) + P(q-d),$$

or

$$\frac{dl}{dt} + ql < S(r+q) < k(r+q) = m,$$

Where

$$q = \min(q_2 E, d),$$

So

$$l = \frac{m}{q} + Ae^{-qt},$$

Thus for

$$t \longrightarrow \infty,$$

$$0 \leq l \leq \frac{m}{q},$$

So, all the solutions of the system (2.1) which start in $R^3_+$ are bounded.

**Predators interaction with susceptible prey**

By field study, it is verified that the interaction of the predator with infected prey is 31 times higher than the interaction of predator with uninfected prey (Lafferty and Morris, 1996). Using this experimental fact, it is quite natural to ignore the interaction of predator with susceptible prey, and so the model (2.1) simplifies to:

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{k}\right) - \frac{\lambda SI}{a+\alpha I} - q_1 ES, \\ \frac{dI}{dt} &= \frac{\lambda SI}{a+\alpha I} - \frac{mI^2 P}{I+S} - \mu I - q_2 EI, \\ \frac{dP}{dt} &= \frac{mI^2 P}{I+S} - dP. \end{aligned} \tag{4.1}$$

Here we are assuming that the growth rate of susceptible prey is high to replace the infected prey population otherwise whole prey population will become infected and ultimately predator-prey population will become extinct.

**Analytical solution**

We get three possible biologically meaningful equilibria by setting time derivative to zero.

(i) $\bar{E_0} = (0, 0, 0)$.

(ii) $\bar{E_1} = \left(\frac{k(r - q_1 E)}{r}, 0, 0\right)$, $where \quad r > q_1 E$.

(iii) $\bar{E_2} = (\bar{S}, \bar{I}, 0)$.

By using the second equation of the system (4.1), we get

$$\bar{S} = \frac{a + \alpha \bar{I}}{\lambda}(\mu + q_2 E),$$

and by substituting $\bar{S}$ in the first equation of the system (4.1), we get

$$(r\lambda\alpha + rq_2 E\alpha^2 + r\mu\alpha^2)\bar{I}^2 + (r\lambda a - r\lambda k\alpha + 2rq_2 Ea\alpha + 2r\mu a\alpha + q_1 E\lambda k\alpha + \lambda^2 k)\bar{I} + (-r\lambda ka + rq_2 Ea^2 + r\mu a^2 + q_1 E\lambda ka) = 0.$$

It can be written as

$$e_0 I^2 + e_1 I + e_2 = 0, \tag{5.1}$$

Where

$$\begin{aligned} e_0 &= (r\lambda\alpha + rq_2 E\alpha^2 + r\mu\alpha^2), \\ e_1 &= (r\lambda a - r\lambda k\alpha + 2rq_2 Ea\alpha + 2r\mu a\alpha + q_1 E\lambda k\alpha + \lambda^2 k), \\ e_2 &= (-r\lambda ka + rq_2 Ea^2 + r\mu a^2 + q_1 E\lambda ka). \end{aligned}$$

The equation (5.1) has one real positive root if

a. $e_1 < 0$, and $e_2 < 0$, or

b. $e_1 < 0$, $e_2 > 0$, and $e_1^2 > 4e_0 e_2$, or

c. $e_1 > 0$, and $e_2 < 0$,

(iv) $\bar{E_3} = (\bar{S}, \bar{I}, \bar{P})$,

Where

$$\bar{S} = \frac{m\bar{I}^2 - \bar{I}d}{d}, \tag{5.2}$$

$$\bar{P} = \frac{\bar{I} + \bar{S}}{m\bar{I}}\left(\frac{\lambda\bar{S}}{a + \alpha\bar{I}} - \mu - q_2 E\right), \tag{5.3}$$

and

$$r\left(1 - \frac{\bar{S} + \bar{I}}{k}\right) - \frac{\lambda\bar{I}}{a + \alpha\bar{I}} - q_1 E = 0. \tag{5.4}$$

Using equations (5.2) and (5.4), we get

$$\bar{I}^3(mr\alpha) + \bar{I}^2(mra) + I\left((q_1 E - r)kd\alpha + \lambda kd\right) + \left((q_1 E - r)kda\right) = 0. \tag{5.5}$$

We can verify from the first equation of (4.1) that $r > q_1 E$. Hence, from Descartes sign rule the equation (5.5) has unique positive roots, and so the coexistence equilibrium of the system (4.1) will be unique.

**Stability analysis**

**Stability of the system around the equilibrium points $\bar{E}_0$, $\bar{E}_1$, and $\bar{E}_2$**

It can be easily shown that equilibrium points $\bar{E}_0$, $\bar{E}_1$ are unstable, and $\bar{E}_2$ is neutrally stable. The characteristic equation of the stability matrix is

$$(\xi - (r - q_1 E))(\xi + (\mu + q_2 E))(\xi + d) = 0.$$

Hence the equilibrium $\bar{E}_0$ will be unstable because of one of the eigenvalues
$r - q_1 E > 0$, and other two eigenvalues $-(\mu + q_2 E)$ and $-d$ are negatives.
Stability analysis of the system (4.1) around $\bar{E}_1 = (\bar{S}, 0, 0)$. The stability matrix leads to the characteristic equation

$$(\xi + (\frac{r\bar{S}}{k}))(\xi + (-\frac{\lambda\bar{S}}{a} + \mu + q_2 E))(\xi + d) = 0.$$

Eigenvalues are

$$\xi_1 = -\frac{r\bar{S}}{k} < 0,\ \xi_2 = (\frac{\lambda\bar{S}}{a} - \mu - q_2 E) > 0, \text{ and } \xi = -d < 0.$$

Hence, the equilibrium $\bar{E}_1$ will be unstable.

Stability analysis of the system (4.1)around $\bar{E}_2 = (\bar{S}, \bar{I}, 0)$.
The associated characteristic equation is

$$-\xi\left[\xi^2 + \xi\left(\frac{r\bar{S}}{k} + \frac{\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})^2}\right) + \frac{r\bar{S}}{k}\left(\frac{\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})^2}\right) + \frac{\lambda r\bar{S}\bar{I}}{k(a+\alpha\bar{I})} + \frac{\lambda^2 a\bar{S}\bar{I}}{k(a+\alpha\bar{I})^3}\right] = 0.$$

Eigenvalues are

$$\xi_1 = 0$$

and

$$\xi_{2,3} = -\left(\frac{r\bar{S}}{k} + \frac{\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})^2}\right) \pm \sqrt{\left(\frac{r\bar{S}}{k} + \frac{\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})}\right)^2 - 4\frac{r\bar{S}}{k}\left(\frac{\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})^2} + \frac{\lambda r\bar{S}\bar{I}}{k(a+\alpha\bar{I})} + \frac{\lambda^2 a\bar{S}\bar{I}}{(a+\alpha\bar{I})^3}\right)} < 0.$$

As a result, the equilibrium $\bar{E}_2$ will be neutrally stable.

**Stability of the system around the equilibrium point $\bar{E}_3$**

The stability matrix about the co-existence equilibrium $\bar{E}_3$ of the system (4.1) is given by

$$\begin{pmatrix} A-\xi & B & 0 \\ C & D-\xi & Q \\ F & G & -\xi \end{pmatrix}. \qquad (6.1)$$

This leads to the characteristic equation

$$\xi^3 + \xi^2(-D-A) + \xi(AD - GQ - BC) + (AGQ - BQF) = 0, \qquad (6.2)$$

With

$$A = \frac{-r\bar{S}}{k}; \quad B = \frac{-r\bar{S}}{k} - \frac{\lambda\bar{S}a}{(a+\alpha\bar{I})^2}; \quad C = \frac{\lambda\bar{I}}{(a+\alpha\bar{I})} + \frac{m\bar{I}^2\bar{P}}{(\bar{I}+\bar{S})^2};$$
$$F = \frac{-m\bar{I}^2\bar{P}}{(\bar{I}+\bar{S})^2}; \quad D = \frac{-\lambda\alpha\bar{S}\bar{I}}{(a+\alpha\bar{I})^2} - \frac{m\bar{I}\bar{P}\bar{S}}{(\bar{I}+\bar{S})^2}; \quad G = \frac{m\bar{I}\bar{P}}{(\bar{I}+\bar{S})^2}(\bar{I}+2\bar{S}); \text{ and}$$
$$Q = \frac{-m\bar{I}^2}{(\bar{I}+\bar{S})^2}. \qquad (6.3)$$

Equation (6.2) can be written in the form

$$\xi^3 + a_1\xi^2 + a_2\xi + a_3 = 0, \qquad (6.4)$$

where

$$a_1 = -D - A > 0,$$
$$a_2 = AD - GQ - BC > 0,$$
$$a_3 = AGQ - BQF > 0.$$

The Routh-Hurwitz stability criteria for the third order system is (a) $a_1 > 0$; $a_3 > 0$,
(b) $a_1 a_2 > a_3$,
Hence, the coexistence equilibrium will be locally stable to small perturbations if
$a_1 a_2 > a_3$,
or

$$-AD^2 + DGQ + DBC - A^2D + ABC + BQF > 0.$$

Now,

$$-AD^2+DGQ+DBC-A^2D+ABC+BQF=$$

$$\frac{r\lambda^2\alpha^2\bar{S}^3\bar{I}^2}{kH^4}+\frac{2mr\lambda\alpha\bar{P}\bar{S}^3\bar{I}^2}{kH^2L^2}+\frac{rm^2\bar{I}^2\bar{P}^2\bar{S}^3}{kL^4}+\frac{m^2\lambda\alpha\bar{S}\bar{P}\bar{I}^5}{L^3H^2}+$$
$$\frac{m^3\bar{I}^5\bar{P}^2\bar{S}}{L^5}+\frac{2m^2\alpha\lambda\bar{I}^4\bar{P}\bar{S}^2}{L^3H^2}+\frac{2m^3\bar{I}^4\bar{P}^2\bar{S}^2}{L^5}+\frac{r\alpha\lambda^2\bar{S}^2\bar{I}^2}{kH^3}+$$
$$\frac{rm\lambda\bar{I}^2\bar{P}\bar{S}^2}{kL^2H}+\frac{a\alpha\lambda^3\bar{S}^2\bar{I}^2}{H^5}+\frac{ma\lambda^2\bar{S}^2\bar{I}^2\bar{P}}{L^2H^3}+\frac{rm\lambda\bar{I}^3\bar{P}\bar{S}^2}{kL^2H^2}+$$
$$\frac{rm^2\bar{I}^3\bar{P}^2\bar{S}^2}{kL^4}+\frac{ma\alpha\lambda^2\bar{S}^2\bar{I}^3\bar{P}}{L^2H^4}+\frac{a\lambda m^2\bar{I}^3\bar{P}^2\bar{S}^2}{L^4H^2}+\frac{r^2\lambda\alpha\bar{S}^3\bar{I}}{k^2H^2}+ \tag{6.5}$$
$$\frac{mr^2\bar{I}\bar{P}\bar{S}^3}{k^2L^2}+\frac{r^2\lambda\bar{S}^2\bar{I}}{k^2H}+\frac{r^2m\bar{I}^2\bar{P}\bar{S}^2}{K^2L^2}+\frac{ra\lambda^2\bar{I}\bar{S}^2}{kH^3}+$$
$$\frac{arm\lambda\bar{I}^2\bar{P}\bar{S}^2}{kL^2H^2}-\frac{rm^2\bar{I}^4\bar{P}\bar{S}}{kL^3}-\frac{\lambda m^2a\bar{I}^4\bar{P}\bar{S}}{H^2L^3},$$

where
$L=\bar{I}+\bar{S}$, and $H=a+\alpha\bar{I}$.
There are only two negative terms in (6.5). To show that sum of all terms in (6.5) is positive, we compare negative terms with positive terms.

First, take

$$-\frac{\lambda m^2a\bar{I}^4\bar{P}\bar{S}}{H^2L^3}+\frac{m^2\lambda\alpha\bar{S}\bar{P}\bar{I}^5}{L^3H^2}+\frac{2m^2\alpha\lambda\bar{I}^4\bar{P}\bar{S}^2}{L^3H^2} \tag{6.6}$$

$$=\frac{m^2\lambda\bar{I}^4\bar{S}\bar{P}}{L^3H^2}\left(-a+\alpha\bar{I}+2\alpha\bar{S}\right)>0, \quad \text{if}$$

$$\bar{I}+2\bar{S}>\frac{a}{\alpha}.$$

Now choose the terms

$$-\frac{rm^2\bar{I}^4\bar{P}\bar{S}}{kL^3}+\frac{2m^3\bar{I}^4\bar{P}^2\bar{S}^2}{L^5}+\frac{m^3\bar{I}^5\bar{P}^2\bar{S}}{L^5} \tag{6.7}$$

$$=\frac{m^2\bar{I}^4\bar{P}\bar{S}}{L^3}\left(\frac{-r}{k}+\frac{2m\bar{P}\bar{S}}{L^2}+\frac{m\bar{I}\bar{P}}{L^2}\right)$$

$$=\frac{m^2\bar{I}^4\bar{P}\bar{S}}{L^3}\left(\frac{-r}{k}+\frac{m\bar{P}\bar{S}}{L^2}+\frac{m\bar{P}(\bar{I}+\bar{S})}{L^2}\right)>0,$$

if

$$r<\frac{mk\bar{P}}{L}\left(1+\frac{\bar{S}}{L}\right),$$

where
$k>L$ and $m\bar{P}>1$.

The natural growth rate of the fish population will be less than unity because the growth rate more than on is true for bacteria and viruses. $m\bar{P}>1$ because the number of a newly born predator by consuming infected prey will be more than one.

**Theorem 3** *Suppose* $\bar{E}3=(\bar{S},\bar{I},\bar{P})$ *exists and*

$$\bar{I}+2\bar{S}>\frac{a}{\alpha},$$

then coexistence equilibrium $\bar{E}3$ will be asymptotically stable.

**Hopf bifurcation analysis**

Taking $\mu$ as the bifurcation parameter we discussed the Hopf bifurcation for the system (4.1). The characteristic equation (6.2) which arise for coexistence equilibrium have positive coefficients $a_1$, $a_2$, $a_3$ and have two purely imaginary roots if $a_1a_2=a_3$ for some value of $\mu$ ( say $\mu=\bar{\mu}$ ) .
A unique µ will exist to satisfy the equation a1a2 = a3. Therefore, there is only one value of µ at which we have Hopf bifurcation. The characteristic equation (6.2) cannot have real positive roots in the neighborhood of $\bar{\mu}$. For $\mu=\bar{\mu}$ the characteristic equation (6.2) can be written as

$$(\xi^2+a_2)(\xi+a_1)=0, \tag{7.1}$$

which had tree roots

$$\xi_1=i\sqrt{a_2}, \quad \xi_2=-i\sqrt{a_2}, \quad \xi_3=-a_1.$$

The general roots are in the form

$$\xi_1(\mu)=p(\mu)+iq(\mu),$$
$$\xi_2(\mu)=p(\mu)-iq(\mu),$$
$$\xi_3(\mu)=-a_1(\mu).$$

Using Hopf bifurcation theorem [**?**] to (4.1), we will use the transversality condition.

$$\Re\left(\frac{d\xi_k}{d\mu}\right)_{\mu=\bar{\mu}} \neq 0,\ \ k=1,2. \tag{7.2}$$

Using

$$\xi_k(\mu) = p(\mu)+iq(\mu)$$

into the equation ( 6.2) and calculating the derivative, we get

$$\begin{aligned} &R(\mu)p'(\mu) - S(\mu)q'(\mu) + T(\mu) = 0, \\ &S(\mu)p'(\mu) + R(\mu)q'(\mu) + U(\mu) = 0, \end{aligned} \tag{7.3}$$

Where

$$\begin{aligned} R(\mu) &= 3p^2(\mu) + 2a_1(\mu)p(\mu) + a_2(\mu) - 3q^2(\mu), \\ S(\mu) &= 6(\mu)q(\mu) + 2a_1(\mu)q(\mu), \\ T(\mu) &= p^2(\mu)a_1'(\mu) + a_2'(\mu)p(\mu) + a_3'(\mu) - a_1'(\mu)q^2(\mu), \\ U(\mu) &= 2p(\mu)q(\mu)a_1' + a_2'(\mu)q(\mu), \end{aligned} \tag{7.4}$$

if $SU + RT\ f= 0$ at $\mu = \overline{\mu}$, then

$$\Re\left(\frac{d\xi_k}{d\mu}\right)_{\mu=\bar{\mu}} = -\frac{SU+RT}{2(R^2+S^2)} \neq 0. \tag{7.5}$$

Now from equation (7.4)

$$SU + RT = 2a_2\left(a_1a_2' + a_1'a_2 - a_3\right)\ at\ \ \mu = \bar{\mu}$$

So

$SU + RT\ f= 0$

If

$$a_1a_2' + a_1'a_2 - a_3 \neq 0,$$

or

$$-2ADD' - A^2D' + DG'E + DBC' + ABC' + GED' + BCD' + BEF' \neq 0, \tag{7.7}$$

or

$$\begin{aligned} &\frac{2\lambda r\alpha m\bar{S}^3\bar{I}^2}{kL^2H^2} + \frac{2rm^2\bar{I}^2\bar{S}^3\bar{P}}{kL^4} + \frac{r^2m\bar{I}\bar{S}^3}{k^2L^2} + \frac{\lambda\alpha m^2\bar{I}^5\bar{S}}{H^2L^3} + \\ &\frac{2\lambda\alpha m^2\bar{I}^4\bar{S}^2}{H^2L^3} + \frac{m^3\bar{I}^5\bar{S}\bar{P}}{L^5} + \frac{2m^3\bar{I}^4\bar{S}^2\bar{P}}{L^5} + \frac{m\lambda\alpha r\bar{S}^2\bar{I}^3}{kL^2H^2} + \\ &\frac{m\lambda^2\alpha a\bar{I}^3\bar{S}^2}{L^2H^4} + \frac{m^2r\bar{I}^3\bar{S}^2\bar{P}}{kL^4} + \frac{m^2\lambda a\bar{I}^3\bar{S}^2\bar{P}}{L^4H^2} + \frac{mr^2\bar{I}^2\bar{S}^2}{k^2L^2} + \\ &\frac{\lambda mra\bar{S}^2\bar{I}^2}{kL^2H^2} + \frac{m^3\bar{S}\bar{I}^5\bar{P}}{L^5} + \frac{2m^3\bar{S}^2\bar{I}^4\bar{P}}{L^5} + \frac{mr\lambda\bar{S}^2\bar{I}^2}{kL^2H} + \\ &\frac{m^2r\bar{I}^3\bar{S}^2\bar{P}}{kL^4} + \frac{m\lambda^2a\bar{I}^2\bar{S}^2}{H^3L^2} + \frac{m^2\lambda a\bar{I}^3\bar{S}^2\bar{P}}{H^2L^4} - \frac{m^2r\bar{S}\bar{I}^4}{kL^3} - \frac{m^2\lambda a\bar{S}\bar{I}^4}{H^2L^3}. \end{aligned} \tag{7.8}$$

where last two terms are negative, and the rest of the terms in (7.8) are positive. We are not considering the term

$$\frac{d\bar{P}}{d\mu}$$

in (7.8) which in common in all terms. Since $d\mu$ at a point of Hopf bifurcation $a_1a_2 = a_3$. So we replace last two-term of (7.8) by last two terms of (6.5) by multiplying by $\overline{P}$, we get

$SU + RT\ f= 0,$

If

$$\begin{aligned} &\frac{m^2r\bar{S}^2\bar{I}^2\bar{P}^2}{kL^3} + \frac{m^3\bar{S}\bar{I}^4\bar{P}^2}{L^4} + \frac{m^3\bar{S}^2\bar{I}^4\bar{P}^2}{L^5} + \frac{m^2\lambda a\bar{S}^2\bar{I}^3\bar{P}^2}{L^4H^2} \\ &> \frac{\lambda^2\alpha^2r\bar{S}^3\bar{I}^2}{kH^4} + \frac{\lambda^3\alpha a\bar{S}^2\bar{I}^2}{H^5} + \frac{r^2\lambda\bar{S}^2\bar{I}}{k^2H} + \frac{\lambda r^2\alpha\bar{S}^3\bar{I}}{k^2H^2} + \frac{\lambda^2r\bar{S}^2\bar{I}}{kH^2}. \end{aligned} \tag{7.9}$$

**Theorem 4** Suppose $\overline{E}_3 = (\overline{S}, \overline{I}, \overline{P})$ exists and inequality (7.9) satisfies, then the system (4.1) exhibits a Hopf bifurcation for a suitable value of μ in the neighborhood of $\overline{\mu}$.

(7.6)

We used a Runge-Kutta -Fehlberg fourth-order to fifth order method to integrate numerically the system (4.1). For simulation, all used computerically generated hypothetical parameters given in (Table 1). The initial values are all slightly perturbed equilibrium values. The numerical results show that there are two Hopf bifurcations for this system (4.1) where stable behaviour changes to unstable as the parameter $\mu$ *is* varied. The first bifurcation point is approximately when $\mu$ = 0.1215 and the second bifurcation point is approximately when $\mu$ = 0.1446 as illustrated in (Figure 1) which contains a plot of $a_1a_2$ $a_3$ as a function of the parameter $\mu$. Figure 2 shows a stable solution for the system when $\mu$ = 0.121 while (Figure 3) shows an unstable solution when the value $\mu$ = 0.122 is used. Figure 4 shows an unstable solution for the system when $\mu$ = 0.144 while (Figure 5) shows a stable solution when the value $\mu$ = 0.145 is used.

**Predator interact with both infected and uninfected prey**

**Table 1**. Representative set of parameter values used for model equation (4.1).

| $q_1$ | $q_2$ | $E$ | $r$ | $k$ | $m$ | $a$ | $\alpha$ | $\lambda$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.0038 | 0.03 | 0.5 | 0.1 | 200 | 0.5 | 400 | 2 | 13.96 | 0.5 |

$q_1$, $q_2$ , $E$ , $r$ , $m$ , $\alpha$, $\lambda$ and $d$ have a units of "per day", while $a$, $K$, $S$, $I$, and $P$ have units of "number per unit area."

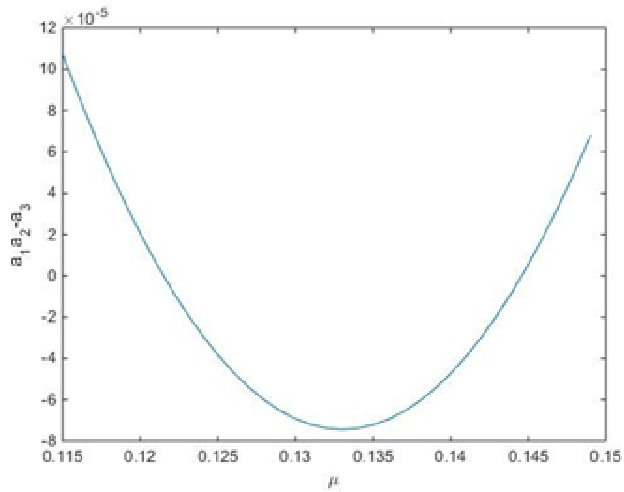


**Figure 1**. Plot of $a_1a_2 - a_3$ as a function of the parameter $\mu$.

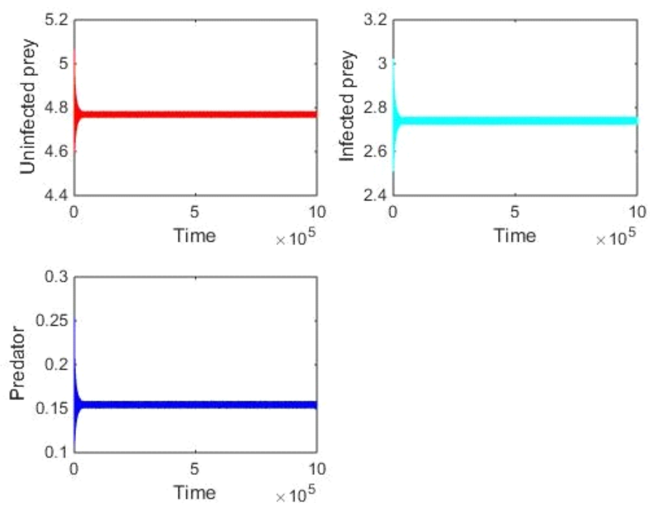


**Figure 2**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.121.

In this section we are studying a generalized model (2.1) where predator consumes both infected and uninfected prey through the interaction of predator with infected prey will be much higher than uninfected prey. The reason behind this is that infected prey due to disease become slow and at the time of death come to the surface of the sea and become more vulnerable to a predator.

**Stability of the system around the equilibrium points $\hat{E}_0$, $\hat{E}_1$, and $\hat{E}_2$**

It can be easily shown that equilibrium points $\hat{E}_0$, $\hat{E}_1$ are unstable and $\hat{E}_2$ is neutrally stable. Stability analysis of the system (2.1) around $\hat{E}_0$ = (0, 0, 0). The characteristic equation of the stability matrix is

$(\xi - (r - q_1E))(\xi + (\mu + q_2E)))(\xi + d) = 0.$

Hence the equilibrium $\bar{E}_0$ will be unstable because one of the eigenvalues $r - q_1E > 0$ and other two eigenvalues $-(\mu + q_2E)$ and $-d$ are negative.

**Stability analysis of the system (2.1) around $\hat{E}_1 = (\hat{S}, 0, 0)$**

The stability matrix leads to the characteristic equation

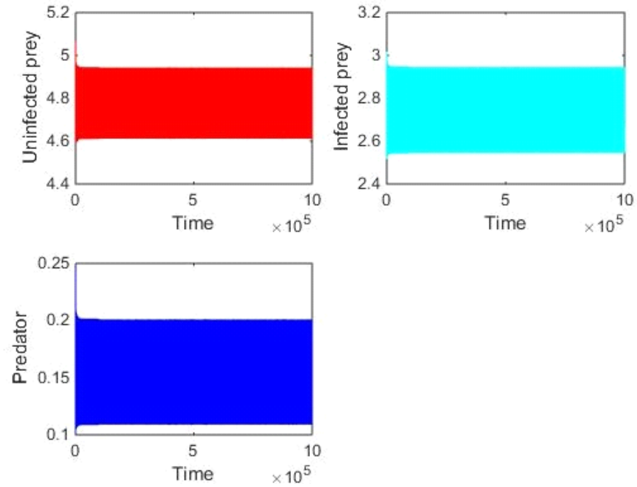


**Figure 3**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.122.

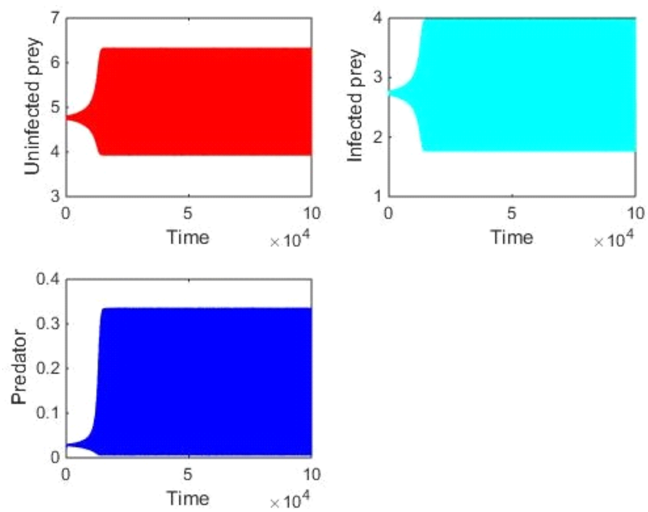


**Figure 4**. Hopf bifurcation with respect to $\mu$ when $\mu$= 0.144.

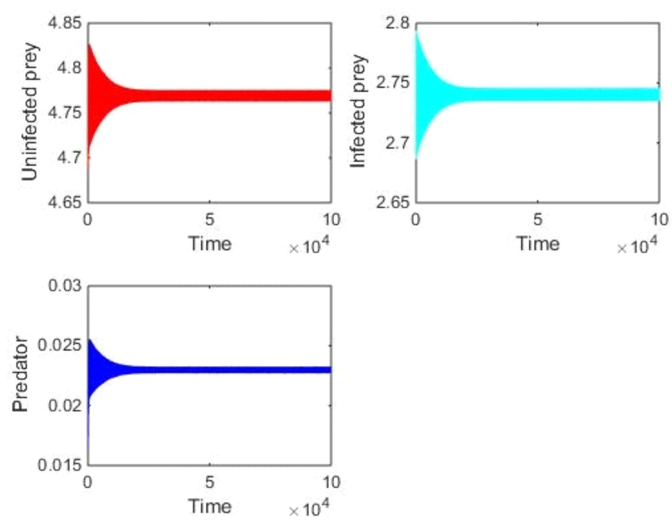


**Figure 5**. Hopf bifurcation with respect to $\mu$ when $\mu$= 0.145.

$$\left(\xi+\frac{r\hat{S}}{k}\right)\left(\xi+(-\frac{\lambda\hat{S}}{a}+\mu+q_2E)\right)\left(\xi+(-n\hat{S}+d)\right)=0.$$

Eigenvalues are

$$\xi_1=-\frac{r\hat{S}}{k}<0,\ \xi_2=(\frac{\lambda\hat{S}}{a}-\mu-q_2E),\ \text{and}\ \xi_3=$$

If $n\hat{S}<d$. Hence, the equilibrium $\bar{E}_1$ will be unstable.

**Stability analysis of the system (2.1) around $\hat{E}_2=(\hat{S},0,\hat{P})$**

The associated characteristic equation is

$$\left(\xi-(\frac{\lambda\hat{S}}{a}-\mu-q_2E)\right)\left(\xi^2+\frac{r\hat{S}}{k}\xi-n^2\hat{S}\hat{P}(1-2\hat{S})\right)=0.$$

Eigenvalues are

$$(\xi_1=(\frac{\lambda\hat{S}}{a}-\mu-q_2E)>0),\ \text{and}$$

$$\xi_{2,3}=-\left(\frac{r\hat{S}}{k}\right)\pm\sqrt{(\frac{r\hat{S}}{k})^2+4(n^2\hat{S}\hat{P}(1-2\hat{S})}.$$

As a result, the equilibrium $\bar{E}\ 2$ will be unstable stable.

**Equilibria**

The system (2.1) has the following equilibria:

(i) $\hat{E}_0=(0,0,0)$.

(ii) $\hat{E}_1=\left(\frac{k(r-q_1E)}{r},0,0\right)$, where $r>q_1E$.

(iii) $\hat{E}_2=\left(\hat{S},0,\hat{P}\right)$, where $\hat{S}=\frac{d}{n}$, $\hat{P}=\frac{1}{n}\left(r-\frac{rd}{kn}-q_1E\right)$ and $r>\frac{rd}{kn}+q_1E$.

(iv) The coexisting equilibrium $\hat{E}_3=\left(\hat{S},\hat{I},\hat{P}\right)$, where

$$\hat{S}=\frac{d(1+\bar{x})\bar{x}}{n\bar{x}^2+m},\quad \hat{I}=\frac{d(1+\bar{x})}{n\bar{x}^2+m},\ \text{and}\ \hat{P}=\frac{1+\bar{x}}{m}\left(\frac{\lambda\hat{S}}{a+\alpha\hat{I}}-\mu-q_1E\right),$$

or equivalently

$$\hat{P}=\frac{1+\bar{x}}{n\bar{x}}\left(r(1-\frac{\hat{S}+\hat{I}}{k})-\frac{\lambda\hat{I}}{a+\alpha\hat{I}}-q_1E\right), \qquad (9.1)$$

equating above two values of $\hat{P}$, we obtain

$$\bar{x}^5(\lambda dkn)+\bar{x}^4(\lambda dkn^2-Gmbn^2k-mrbn(nk-d)+q_1Emk\alpha dn^2)+$$

$$\bar{x}^3(2mn\lambda dk-Gnk\alpha dm-\alpha dmr(nk-d)+2bndmr+q_1Emk\alpha dn)+$$

$$\bar{x}^2(2mn\lambda dk-Gnk(L+mbn)-L(nk-d)mr+2\alpha d^2mr-Mbnmr+Lnq_1Emk+m^2bndmr)+$$

$$\bar{x}(m^2\lambda dk-Gnkm\alpha d+2dLmr-M\alpha dmr+q_1Em^2k\alpha d)+$$

$$(9.2)$$

$$(\lambda dkm^2-GnkmL-mrML+q_1Em^2kL)=0,$$

where $\bar{x}=\frac{\hat{S}}{\hat{I}}$, is the real positive root of the equation (9.2), $M=mk-d$, $G=\mu+q_2E$, and $L=am+\alpha d$.

**Stability**

The stability matrix of coexisting equilibrium is

$$\begin{pmatrix} L-\xi & M & N \\ P & Q-\xi & R \\ U & T & -\xi \end{pmatrix}. \qquad (9.3)$$

The characteristic equation associated with the co-existing equation is

$$\xi^3-\xi^2(L+Q)+\xi(LQ-RT-MP-NU)+(LRT-MRU-NPT+NUQ)=0, \qquad (9.4)$$

Where

$$L=\frac{-r\hat{S}}{k}-\frac{n\hat{S}\hat{P}\hat{I}}{A^2}<0;\quad M=\frac{-r\hat{S}}{k}+\frac{\lambda\hat{S}\hat{I}\alpha}{B^2}-\frac{\lambda\hat{S}}{B}+\frac{n\hat{S}^2\hat{P}}{A^2};\quad N=\frac{-n\hat{S}^2}{A}<0;$$

$$P=\frac{\lambda\hat{I}}{B}+\frac{m\hat{I}^2\hat{P}}{A^2}>0;\quad Q=\frac{-\lambda\hat{S}\hat{I}\alpha}{B^2}-\frac{m\hat{I}\hat{P}\hat{S}}{A^2}<0;\quad R=-\frac{m\hat{I}^2}{A}<0;$$

$$U=\frac{\hat{S}\hat{P}n}{A^2}(\hat{S}+2\hat{I})-\frac{m\hat{I}^2\hat{P}}{A^2};\quad T=\frac{m\hat{I}\hat{P}}{A^2}(2\hat{S}+\hat{I})-\frac{n\hat{S}^2\hat{P}}{A^2};$$

where, $A=\hat{S}+\hat{I}$; $B=a+\alpha\hat{I}$.

$$\text{If}\ \frac{\hat{S}^2}{\hat{I}^2}=\frac{m}{n},\ \text{then}\ U,\ \text{and}\ T>0.$$

Using Routh-Hurwitz stability criteria of the third order system, we can say that the coexistence equilibria is stable if

**Table 2**. Representative set of parameter values used for model equation (2.1).

| $q_1$ | $q_2$ | $E$ | $r$ | $k$ | $n$ | $m$ | $a$ | $\alpha$ | $\lambda$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0038 | 0.03 | 0.5 | 0.1 | 200 | 0.01 | 0.5 | 400 | 2 | 13.96 | 0.5 |

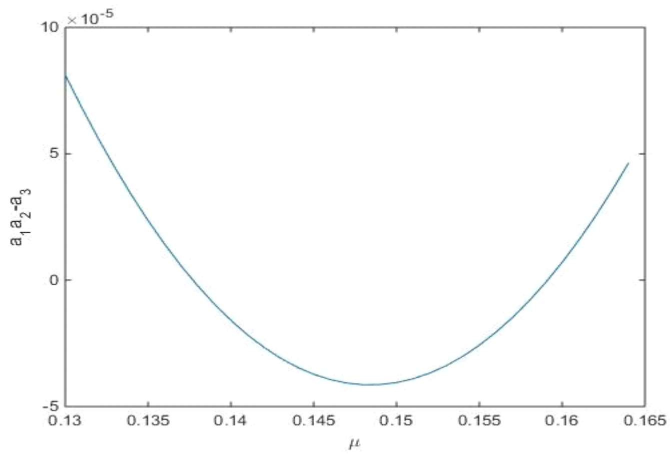


**Figure 6**. Plot of $a_1a_2 - a_3$ as a function of the parameter $\mu$.

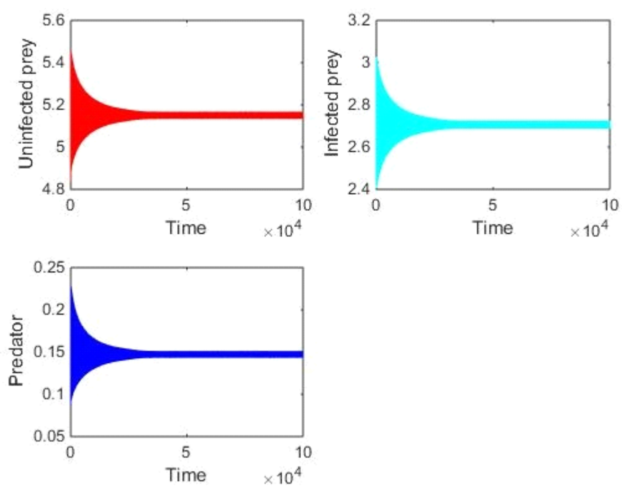


**Figure 7**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.137.

$$\frac{\lambda\hat{S}\hat{I}\alpha}{B^2}+\frac{n\hat{S}^2\hat{P}}{A^2}>\frac{r\hat{S}}{k}+\frac{\lambda\hat{S}}{B}, \qquad (9.5)$$

and

$$LNS+Q(RT-L^2-LQ)>M(LP+PQ+RS)+NPT. \qquad (9.6)$$

### Numerical results

The system (2.1) has been integrated numerically using a Rung-Kutta-Fehlberg fourth-fifth order method. Table 2 contains a representative set of values used for the hypothetical parameters in the simulation. The numerical results show that there are two Hopf bifurcations for this system (2.1) where stable behaviour changes to unstable as the parameter $\mu$ is varied. The first bifurcation point is approximately when $\mu$ = 0.1378 and the second bifurcation point is approximately when $\mu$ = 0.1592 as illustrated in (Figure 6) which contains a plot of $a_1a_2$ $a_3$ as a function of the parameter $\mu$. Figure 7 shows a stable solution for the system when $\mu$ = 0.137 while (Figure 8) shows an unstable solution when the value $\mu$ = 0.138 is used. Figure (9) shows an unstable solution for the system

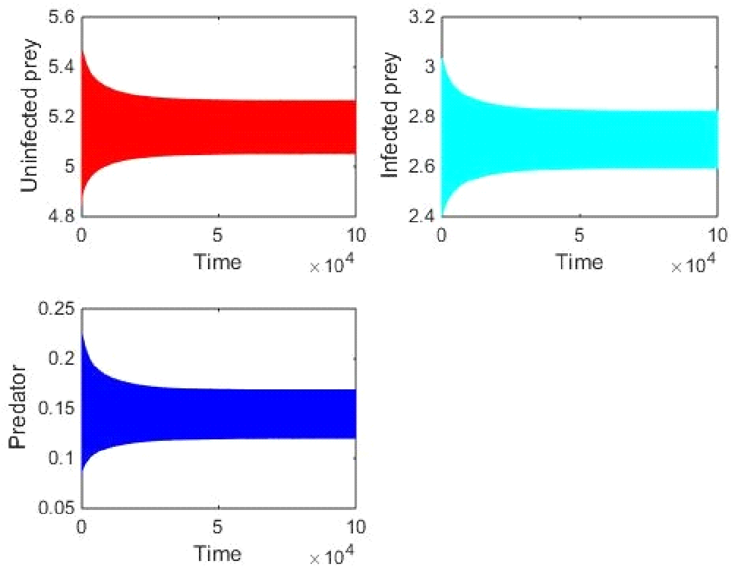


**Figure 8**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.138.

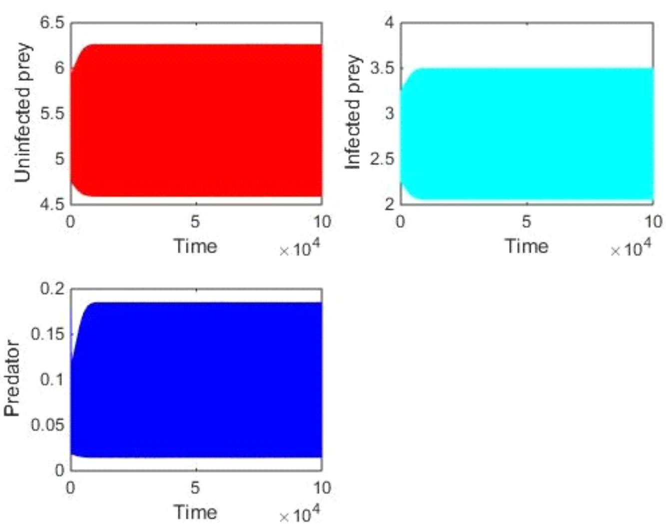


**Figure 9**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.159.

when $\mu$ = 0.159 while (Figure 10) shows a stable solution when the value $\mu$ = 0.16 is used.

### Asymptotic stability of co-existing equilibrium

**Theorem 5** *If the uninfected and infected prey population satisfies the equation of a straight line S = mI where*

$$m = \frac{\hat{S}}{\hat{I}},$$

then co-existing equilibrium of the (2.1) will be asymptotically stable.

### Proof

Let us consider a positive definite function

$$V(S,I,P) = (S-\hat{S}) - \hat{S}\ln\left(\frac{S}{\hat{S}}\right) + (I-\hat{I}) - \hat{I}\ln\left(\frac{I}{\hat{I}}\right) + (P-\hat{P}) - \hat{P}\ln\left(\frac{P}{\hat{P}}\right), \quad (11.1)$$

the bounded region *D*. Since *S, I,* and *P* are positives in the bounded region *D*. The derivative of the equation (11.1) along the solution of the system of equation (2.1), we get

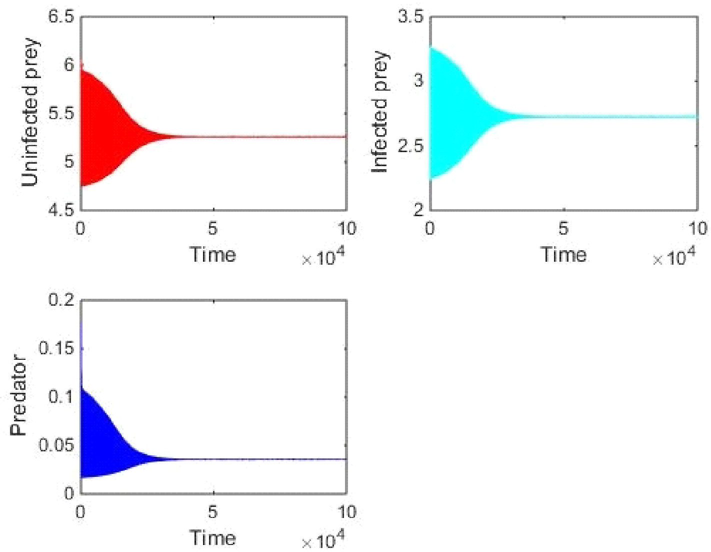


**Figure 10**. Hopf bifurcation with respect to $\mu$ when $\mu$ = 0.16.

$$\frac{dV}{dt}=\left(\frac{S-\hat{S}}{S}\right)\frac{dS}{dt}+\left(\frac{I-\hat{I}}{I}\right)\frac{dI}{dt}+\left(\frac{P-\hat{P}}{P}\right)\frac{dP}{dt},$$
$$=(S-\hat{S})\left(r\left(1-\frac{S+I}{k}\right)-\frac{\lambda I}{b+\alpha I}-\frac{nSP}{I+S}-q_1E\right)$$
$$+(I-\hat{I})\left(\frac{\lambda S}{b+\alpha I}-\frac{mIP}{I+S}-\mu-q_2E\right) \qquad (11.2)$$
$$+(P-\hat{P})\left(\frac{nS^2}{I+S}+\frac{mI^2}{I+S}-d\right).$$

At equilibrium,

$$r-q_1E=r\left(\frac{\hat{S}+\hat{I}}{k}\right)+\frac{\lambda\hat{I}}{a+\alpha\hat{I}}-\frac{n\hat{S}\hat{P}}{\hat{I}+\hat{S}}$$
$$u+q_2E=\frac{\lambda\hat{S}}{a+\alpha\hat{I}}-\frac{m\hat{I}\hat{P}}{\hat{I}+\hat{S}},$$

and

$$d=\frac{n\hat{S}^2}{\hat{S}+\hat{I}}+\frac{m\hat{I}^2}{\hat{I}+\hat{S}},$$

so

$$\frac{dV}{dt}=\frac{-r}{k}(S-\hat{S})^2-\frac{r}{k}(S-\hat{S})(I-\hat{I})+\frac{\lambda\alpha}{(a+\alpha I)(a+\alpha\hat{I})}(S\hat{I}-I\hat{S})(I-\hat{I})$$
$$+\frac{n}{(I+S)(\hat{S}+\hat{I})}(S\hat{P}-\hat{S}P)(I\hat{S}-S\hat{I})+\frac{m}{(I+S)(\hat{S}+\hat{I})}(I\hat{P}-\hat{I}P)(S\hat{I}-I\hat{S}).$$

Let

$S\hat{I}=I\hat{S}$,

or

$$\frac{S}{I}=\frac{\hat{S}}{\hat{I}},$$

Then

$$\frac{dV}{dt}=\frac{-r}{k}(S-\hat{S})^2-\frac{r}{k}(S-\hat{S})(I-\hat{I})+\frac{\lambda\alpha}{(a+\alpha I)(a+\alpha\hat{I})}(S\hat{I}-I\hat{S})(I-\hat{I}),$$

and we know that,

$$\begin{aligned}(S\hat{I}-I\hat{S})(I-\hat{I})&=SI-S\hat{I}-I\hat{S}+\hat{S}\hat{I},\\&=SI-2S\hat{I}+\hat{S}\hat{I},\\&=(\sqrt{SI}-\sqrt{\hat{S}\hat{I}})^2.\end{aligned}$$

So

$$\frac{dV}{dt} = \frac{-r}{k}(S - \hat{S})^2 - (\sqrt{SI} - \sqrt{\hat{S}\hat{I}})^2 < 0.$$

Hence, $\hat{E}_3 = (\hat{S}, \hat{I}, \hat{P})$ will be asymptotically stable for $S = mI$

Where

$$m = \frac{\hat{S}}{\hat{I}},$$

for all $t \geq 0$.

## Conclusion

We studied the dynamical behaviour of a three-dimensional deterministic predator-prey model consisting of three nonlinear ordinary differential equations corresponding to uninfected and, infected prey populations, and a predator. The predator can feed on other kinds of prey, but instead of choosing individuals at random the predator consumes a member of either the uninfected or infected prey population proportional to their abundance. The predator feeds preferentially on the most numerous prey species, and hence upon reduction of the number of infected prey due to heavy predation, the predator begins to target uninfected prey. This behaviour is termed predator switching. In the first model, we assume that predators prey preferentially on infected prey because it is verified by numerous field studies that predators catch infected prey 31 times more often compared with uninfected prey. The second model deals with the situation where predators prey on both uninfected and infected prey according to their numerical superiority. In this model, the predator is more likely to consume infected prey rather than uninfected prey, due to infected prey becoming slow and weak and coming to the surface of the vegetation for oxygen. These models have been investigated using stability, Hopf bifurcation, and numerical analysis.

Zero equilibrium in both models is unstable, i.e. the population will likely never be extinct. Axial equilibria $\bar{E}_1$ and $\hat{E}_1$ which contains only uninfected prey will exist if

$$E < \frac{r}{q_1}.$$

$\bar{E}_1$ and $\hat{E}_1$ will exist for all parametric values, but with harvesting $\bar{E}_1$ and $\hat{E}_1$ exists only if the result of harvesting has a lower threshold value depending on the ratio of the growth of uninfected prey and the catchability coefficient of uninfected prey. The predator and prey population will equilibriums in a system (4.1), so it becomes asymptotically stable around the co-existing equilibrium if the ratio of uninfected and infected prey satisfies an equation of a straight line

$$(S = mI, \text{ where } m = (\tfrac{\bar{S}}{\bar{I}})\ ), \text{ where } \tfrac{\bar{S}}{\bar{I}}$$

is the slope of the straight line Hopf bifurcation analysis has been carried out for both models with respect to the parameter $\mu$ (death rate of infected prey), Hopf bifurcation has helped us in finding the existence of a region of instability in the neighborhood of coexisting equilibrium where both uninfected and infected prey species with predators will survive despite undergoing regular fluctuations. However, the conditions of Hopf bifurcation may or may not be satisfied due to a change in the parameters.

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